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In the second discussion Professor Reynolds outlines a method for teaching the mathematics of investment without the use of the geometric series. In view of the recommendation of the National Committee on Mathematical Requirements that this subject should be studied (by election) in the Secondary School, these suggestions will be of interest to others besides college teachers.

I. A Method of Deriving Formulæ for the Expansion of $\sin(x+y)$ and $\cos(x+y)$.

By H. E. Webb, Central High School, Newark, N. J.

For some time past the writer, in beginning with his classes the subject of plane trigonometry, has defined the sine and the cosine of a *number* as lines in a unit circle (the number being measured in linear units on the circumference), while the tangent and cotangent are defined as the ratios of these to each other, and the cosecant and secant as their respective reciprocals. At a later stage these functions are related to ratios between the sides of a right triangle, when the number is less than $\pi/2$, and are regarded as functions of the central angle which intercepts the arc.[1] This procedure has numerous advantages, among which may be mentioned the presentation of a clear idea of the limiting cases, and of the nature of the discontinuity of the four functions last named; an early acquaintance with "radian measure"; an escape from the necessity of two sets of definitions, one for the acute angle and the other for other angles; and the opportunity for presenting clearly the notions of negative angles and of positive angles in excess of 180°.

In line with this general plan, the following development of the expansion of $\sin(x+y)$ and $\cos(x+y)$ is presented, with the apology that it has not, to the knowledge of the writer, previously appeared in printed form.

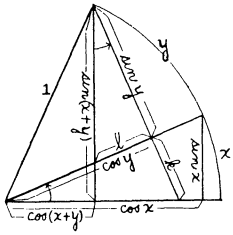

Fig. 1.

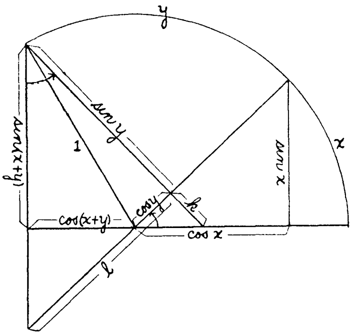

Fig. 2.

The radius of the circle is in each case unity. The notation of the figures is self-explanatory. From similar triangles, two proportions can be set up in each of the cases $\sin(x+y)$ and $\cos(x+y)$. For the case of $\sin(x+y)$,

$$k/\cos y = \sin x/\cos x, \tag{1}$$

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[1] This plan follows an outline of the subject by Professor C. O. Gunther, of Stevens Institute of Technology.

and

$$\sin(x + y)/(k + \sin y) = \cos x/1. \tag{2}$$

These are combined into the equation

$$\begin{aligned}\sin(x + y) &= \cos x\left(\frac{\sin x \cos y}{\cos x} + \sin y\right)\\ &= \sin x \cos y + \cos x \sin y.\end{aligned}$$

Also

$$l/\sin y = \sin x/\cos x, \tag{3}$$

and

$$\cos(x + y)/(\cos y - l) = \cos x/1; \tag{4}$$

whence

$$\begin{aligned}\cos(x + y) &= \cos x\left(\cos y - \frac{\sin x \sin y}{\cos x}\right)\\ &= \cos x \cos y - \sin x \sin y.\end{aligned}$$

In figure 2, since $\cos(x + y)$ is negative and $\cos y$ is less than l, the relation (4) still holds.

The development of $\sin(x - y)$ is obtained by substitution as

$$\cos(\pi/2 - \overline{x - y}) = \cos(\overline{\pi/2 - x} + y),$$

and so for $\cos(x - y)$.

It is assumed above that x and y are in all cases positive and less than $\pi/2$. If either or both are greater, the functions can be transformed by substitutions for x of $\pi \pm x'$ or of $2\pi - x'$, or for y of $\pi \pm y'$ or of $2\pi - y'$, obviating the necessity for any geometric proofs other than those given, to make the formulæ general.

These proofs have the advantage of showing graphically the relative values of the various functions involved.

Remarks by A. A. Bennett.

The Greeks had a strong hold of the notion of what the modern physicist has rediscovered under the title of "dimensions." For Euclid, the product of two line segments was necessarily measured in terms of area. The physical character of the measurements was at every stage so conspicuously in the consciousness that purely arithmetical studies fared ill. The discovery of Analytical Geometry shows, perhaps, the originality of genius more obviously in the following feature than in any other,—Descartes was willing to adopt the consequences of permitting the product of two quantities, each of which was laid off as a segment of a line, to be itself so expressed. In short, he was willing to grant a spatial representation to pure numbers. Since that time, mathematicians have learned much that requires mental contortions. Many obvious theorems are now known to be false and many time-honored terms have been generalized beyond all recognition. Fortunately, however, any convention must stand upon its own merits and vindicate its right to adoption. Are we prepared to accept a convention that makes trigonometry deal with six fundamental quantities, of which sines and cosines are line segments, tangents and cotangents are pure numbers, and secants and cosecants are the reciprocals of line segments—and consequently of dimension minus unity, whatever that may mean? As for the angles on which the whole discussion is conceivably based, why, they do not appear at all; these are all functions of abstract numbers. Is not the relation, $\sin^2 x + \cos^2 x = 1$, worth preserving, when the symbol 1 stands, as it ought, for a pure number? These harsh remarks are, of course, no reflection on the interest offered by the method.

II. THE DERIVATION OF FORMULÆ IN THE MATHEMATICS OF INVESTMENT.

BY C. N. REYNOLDS, JR., West Virginia University.

In teaching the mathematics of investment most of us base the derivation of the formulæ for annuities upon formulæ concerning geometric series which are but faint memories in the minds of most students. Although this links the students' work with their preparatory school work, personal experience in teaching this subject has convinced me that we should make the mathematics of investment an independent mathematical discipline based upon concepts of its own.

In particular I would suggest the adoption of the following fundamental concepts:

(1) The accumulated value of one dollar at the end of n years;
(2) The discounted value of one dollar due at the end of n years;
(3) The present value of a perpetuity of one dollar;

where money is supposed to be worth $100\,i\%$ per annum compounded annually.

I would suggest that the formulæ for these quantities be made to play the rôle of definitions. The definition of the present value of a perpetuity as $1/i$ could be justified by the fact that if a man deposits $1/i$ dollars in a bank which pays $100\,i\%$ interest per annum, he receives in return a perpetuity of one dollar per annum, provided that he and his heirs draw the interest from the bank as soon as it falls due. Similarly the other formulæ could be justified.

Having introduced these definitions, an annuity can now be treated as the difference between two perpetuities. For example, the present value of an annuity of n annual payments of one dollar each will be the present value of a one dollar perpetuity minus the discounted value of a perpetuity beginning after n years, *i.e.*,

$$a_{\overline{n}|} = \frac{1}{i} - \frac{1}{i}(1+i)^{-n} = \frac{1-(1+i)^{-n}}{i}\,.$$

Similarly all the other formulæ of the mathematics of investment can be derived from the three formulæ mentioned above.